Natural solutions of rational Stieltjes moment problems

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Abstract

In the strong or two-point Stieltjes moment problem, one has to find a positive measure on $[0, \infty)$ for which infinitely many moments are prescribed at the origin and at infinity. Here we consider a multipoint version in which the origin and the point at infinity are replaced by sequences of points that may or may not coincide. In the indeterminate case, two natural solutions μ_0 and μ_∞ exist that can be constructed by a limiting process of approximating quadrature formulas. The supports of these natural solutions are disjoint (with possible exception of the origin). The support points are accumulation points of sequences of zeros of even and odd indexed orthogonal rational functions. These functions are recursively computed and appear as denominators in approximants of continued fractions. They replace the orthogonal Laurent polynomials that appear in the two-point case. In this paper we consider the properties of these natural solutions and analyse the precise behaviour of which zero sequences converge to which support points.

Keywords : Stieltjes moment problem, orthogonal polynomials, orthogonal rational functions, continued fractions.
MSC : Primary : 30D15,

Natural solutions of rational Stieltjes moment problems

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Abstract

In the strong or two-point Stieltjes moment problem, one has to find a positive measure on $[0,\infty)$ for which infinitely many moments are prescribed at the origin and at infinity. Here we consider a multipoint version in which the origin and the point at infinity are replaced by sequences of points that may or may not coincide. In the indeterminate case, two natural solutions μ_0 and μ_∞ exist that can be constructed by a limiting process of approximating quadrature formulas. The supports of these natural solutions are disjoint (with possible exception of the origin). The support points are accumulation points of sequences of zeros of even and odd indexed orthogonal rational functions. These functions are recursively computed and appear as denominators in approximants of continued fractions. They replace the orthogonal Laurent polynomials that appear in the two-point case. In this paper we consider the properties of these natural solutions and analyse the precise behaviour of which zero sequences converge to which support points.

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2000 MSC: 30D15, 30E05, 42C05, 44A60

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1. Introduction

The *Stieltjes transform* $S(z,\mu)$ of a finite measure μ on the real line $(-\infty,\infty)$ is here defined as

$$S(z,\mu) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{z-t}. \tag{1.1}$$

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The function $S(z,\mu)$ is holomorphic outside the support $\operatorname{supp}\mu$ of μ and maps the open upper half-plane into the lower closed half-plane.

The *Stieltjes moment problem* for a sequence $\{c_n\}_{n=0}^{\infty}$ of real numbers consists of finding positive measures μ with $\operatorname{supp}\mu \subset [0,\infty)$ such that

$$\int_0^{\infty} t^n \mathrm{d}\mu(t) = c_n \quad \text{for } n = 0,1,2,\ldots. \tag{1.2}$$

An important tool in the treatment of the moment problem is the study of *orthogonal polynomial sequences* $\{\varphi_n\}_{n=0}^{\infty}$ associated with the moments. It is known that the zeros $x_{n,k}$ of φ_n are simple and contained in $(0,\infty)$. When the zeros are numbered such that $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$, then the zeros of φ_n separate the zeros of φ_{n+1} in the sense that

$$x_{n+1,1} < x_{n,1} < x_{n+1,2} < \cdots < x_{n,n} < x_{n+1,n+1}. \tag{1.3}$$

It follows that each sequence $\{x_{n,k}\}_n$ is decreasing to a value $\xi_k \in [0,\infty)$, while each sequence $\{x_{n,n-k}\}_n$ is increasing to a value $\eta_k \in (0,\infty]$, $k = 0,1,2,\ldots$.

The polynomials φ_n are the canonical denominators of a continued fraction (equivalent to a *real Jacobi fraction* or a *Grommer fraction*). The approximants of the continued fraction converge to $S(z,\mu_0)$ for a solution μ_0 of the moment problem. When the moment problem is *indeterminate* (i.e., has more than one solution), the polynomials φ_n suitably normalized converge locally uniformly in the complex plane $\mathbb{C}$ to an entire function Φ_0, which has as its zeros exactly the values ξ_k (while $\eta_k = \infty$ for all k). Furthermore the set $\cup_{k=0}^{\infty}\{\xi_k\}$ constitutes the support of μ_0.

The real Jacobi fraction is the even contraction of another continued fraction (equivalent to a *Stieltjes fraction*). The canonical denominators ψ_n of the odd contraction of this continued fraction have similar properties to those of the φ_n, and give rise to a solution μ_∞ of the moment problem. The approximants of this continued fraction converge to $S(z,\mu_\infty)$. The measures μ_0 and μ_∞ are called *natural solutions* of the Stieltjes moment problem.

For relevant discussions of the Stieltjes moment problem we refer to [3], [9, Ch. 1-2], [16, Sections 42-44], [17, Chapters I,IV,VII].

The *strong* (or *two-point*) *Stieltjes moment problem* for a sequence $\{c_n\}_{n=-\infty}^{\infty}$ of real numbers consists of finding positive measures μ with $\operatorname{supp}\mu \subset [0,\infty)$ such that

$$\int_0^{\infty} t^n \mathrm{d}\mu(t) = c_n \quad \text{for } n = 0,\pm 1,\pm 2,\ldots. \tag{1.4}$$

This concept was introduced in [11], where basic results on existence and uniqueness were obtained. For material relevant to the discussion of strong Stieltjes moment problems below, we refer to [1], [2], [3], [10], [11], [15], [14].

In a way similar to orthogonal polynomials in the classical situation, *orthogonal Laurent polynomials* φ_n play a central role in the theory of strong moment problems. As in the classical case, the zeros of φ_n are simple and contained in $(0,\infty)$, and the zeros of φ_n separate those of φ_{n+1}. When numbered as above, each sequence $\{x_{n,k}\}_n$ decreases to a value $\xi_k \in [0,\infty)$ and each sequence $\{x_{n,n-k}\}_n$ increase to a value $\eta_k \in (0,\infty]$, $k = 0,1,2,\ldots$.

The functions φ_n are the canonical denominators of a continued fraction (equivalent to a *positive Thron fraction*. The even and odd approximants of the continued fraction converge to $S(z,\mu_0)$

and $S(z,\mu_\infty)$, where μ_0 and μ_∞ are solutions of the moment problem. The measures μ_0 and μ_∞ are called *natural solutions* of the strong Stieltjes moment problem. When the problem is *indeterminate*, the sequences $\{\varphi_{2m}\}$ and $\{\varphi_{2m+1}\}$, suitably normalized, converge locally uniformly in $\mathbb{C}\setminus\{0\}$ to distinct functions Φ_0 and Φ_∞. The zeros of Φ_0 and the zeros of Φ_∞ are disjoint. The zeros of Φ_0 together with the origin constitute supp μ_0, while the zeros of Φ_∞ together with the origin constitute supp Φ_∞. The connection between the limits of zero sequences of $\{\varphi_n\}$ and the zeros of Φ_0 and Φ_∞ is radically different from that in the classical situation. Because of the disjointness of the supports of μ_0 and μ_∞ (apart from the origin), we have $\xi_k = 0$ and $\eta_k = \infty$ for all k. The support of μ_0 (resp. μ_∞) consists of all accumulation points of zeros of φ_{2m} (resp. φ_{2m+1}), but in general no results are known concerning *how* the zeros of the orthogonal functions approach the support points of the natural solutions.

When the recurrence relation for the orthogonal Laurent polynomials (equivalently for the positive Thron fraction) has a special form, it has been shown that every sequence of zeros of φ_{2m} (resp. φ_{2m+1}) keeping a fixed position relative to the middle zeros of φ_{2m+1} converges to the zeros of Φ_0 (resp. Φ_∞), i.e., to the support points of μ_0 (resp. μ_∞) in $(0,\infty)$. For these results we refer to [1], [2], [3].

In the present paper we consider certain *rational Stieltjes moment problems* (which in general split in a weaker and a stronger form). Here polynomials or Laurent polynomials are replaced by rational functions with two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of prescribed poles in $(-\infty,0)$ (the $\{\alpha_n\}$ playing the role of the origin and the $\{\beta_n\}$ playing the role of infinity in the two-point case). The *orthogonal rational functions* φ_n associated with the moment problem are the canonical denominators of a continued fraction (a *multipoint Padé continued fraction*). The even and the odd approximants also in this situation converge to $S(z,\mu_0)$ and $S(z,\mu_\infty)$, where the measures μ_0 and μ_∞ are solutions related to the moment problem. These measure μ_0 and μ_∞ are called *natural solutions* for the moment problem.

We show that in the *indeterminate* case the orthogonal rational functions φ_{2m} and φ_{2m+1}, suitably normalized, converge under certain conditions locally uniformly outside the closure of the set of interpolation points $\{\alpha_n\}$, $\{\beta_n\}$. Furthermore the zeros of the limit functions Φ_0 and Φ_∞ constitute together with the origin the support of μ_0 and μ_∞. Finally we prove a result on the convergence of sequences of zeros of φ_{2m} and φ_{2m+1} to the support points of μ_0 and μ_∞, analogous to the result in the two-point situation.

For the theory of rational moment problems and orthogonal rational functions in general and in the Stieltjes situation (support in $[0,\infty)$) in particular we refer to [5], [6], [7], [8] and further references found there.

We shall in this paper mainly use notation as in [8], with some deviations.

2. Rational Stieltjes moment problems

In [8] a general theory of rational moment problems on the non-negative real axis is presented, with underlying assumptions gradually strengthened during the exposition, according to the need in the various arguments. Cf. also [6], [7]. In this and the following sections we repeatedly refer to concepts and results in [8], but we shall consider one and the same setting in the whole paper.

For more general expositions of rational moment problems on the real line and on the unit circle, we refer to [5, Chapter 11], and further references found there.

Our assumptions are the following. Let two sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ of real numbers be given, and set $r_{2m}(z) = (\beta_m - z)$ for $m = 1,2,\ldots$, $r_{2m+1}(z) = (\alpha_{m+1} - z)$ for $m = 0,1,2,\ldots$, $r_0 = 1$, $r_{-1} = 1$. Furthermore we assume there are two numbers α, β in $(-\infty,0]$ such that $-\infty < \beta_j \leq \beta < \alpha \leq \alpha_k \leq 0$ for all j,k. We set $A = \{a \in \mathbb{R} : \alpha_k = a \text{ for some } k\}$, $B = \{b \in \mathbb{R} : \beta_k = b \text{ for some } k\}$, $S = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ where $\mathrm{cl}(\cdot)$ denotes closure of the set. We use $\overline{}$ to denote the complex conjugate.

We may look upon points α_n as playing the role of the origin and the points β_n as playing the role of infinity in the two-point situation. In [8] values $\beta_k = -\infty$ and $\beta = -\infty$ are allowed. Since the development then becomes more cumbersome, we shall not include this possibility in this paper. Thus for purely technical reasons, the theory of strong or two-point moment problems will not be formally covered in this presentation. Even more, we shall assume that all β_k stay away from $-\infty$, i.e., we assume that S is compact.

We set

$$D_n(z) = r_1(z)r_2(z)\cdots r_n(z) \text{ for } n = 1,2,\ldots; \; D_0 = 1. \tag{2.1}$$

Let Π_n denote the space of polynomials of degree at most n, and define the spaces $\mathscr{L}_n$ and $\mathscr{L}$ by

$$\mathscr{L}_n = \left\{ \frac{p(z)}{D_n(z)} : p \in \Pi_n \right\}, \;\; \mathscr{L} = \bigcup_{n=0}^{\infty} \mathscr{L}_n. \tag{2.2}$$

We shall also make use of the space $\mathscr{L} \cdot \mathscr{L}$ consisting of all products of two functions in $\mathscr{L}$. We note that if for every $a \in A$, $b \in B$, we have $\alpha_k = a$, $\beta_k = b$ for infinitely many k, then $\mathscr{L} \cdot \mathscr{L} = \mathscr{L}$. This is in particular the case of each of the sets A, B consists of a finite number of points repeated periodically.

Let M be a positive definite Hermitian linear functional on $\mathscr{L} \cdot \mathscr{L}$. Thus $\overline{M[g]} = M[\overline{g}]$ for every $g \in \mathscr{L} \cdot \mathscr{L}$ and $M[f \cdot \overline{f}] > 0$ for every $f \in \mathscr{L}$, $f \not\equiv 0$. We shall for convenience normalize such that $M[1] = 1$. A measure μ with supp $\mu \subset [0,\infty)$ is said to solve the *rational Stieltjes moment problem on* $\mathscr{L}$ if all functions in $\mathscr{L}$ are absolutely integrable with respect to μ and

$$M[g] = \int_0^{\infty} g(t)\mathrm{d}\mu(t), \text{ for all } g \in \mathscr{L}. \tag{2.3}$$

The measure is said to solve the *rational Stieltjes moment problem on* $\mathscr{L} \cdot \mathscr{L}$ if all functions in $\mathscr{L} \cdot \mathscr{L}$ are absolutely integrable with respect to μ and

$$M[h] = \int_0^{\infty} h(t)\mathrm{d}\mu(t) \text{ for all } h \in \mathscr{L} \cdot \mathscr{L}. \tag{2.4}$$

An equivalent formulation which may make clear the use of the expression *moment problem* can be described as follows. Let $\{\Omega_n\}_{n=0}^{\infty}$ be some basis for $\mathscr{L}$ where $M[\Omega_n]$ is real for all n, for example the basis described in [8, Section 13]. Define the *moments* $c_{j,k}$ as

$$c_{j,k} = M[\Omega_j \Omega_k], \text{ for } j,k = 0,1,2,\ldots. \tag{2.5}$$

Note the symmetry $c_{j,k} = c_{k,j}$. Then μ solves the moment problem on $\mathscr{L}$ if and only if

$$\int_0^\infty \Omega_j(t)\mathrm{d}\mu(t) = c_{j,0} \ \text{ for } \ j = 0,1,2,\ldots, \tag{2.6}$$

and μ solves the moment problem on $\mathscr{L}\cdot\mathscr{L}$ if an only if

$$\int_0^\infty \Omega_j(t)\ \Omega_k(t)\mathrm{d}\mu(t) = c_{j,k} \ \text{ for } \ j,k = 0,1,2,\ldots. \tag{2.7}$$

We denote by $\mathscr{M}_+(\mathscr{L})$ the set of all solutions of the moment problem on $\mathscr{L}$, and by $\mathscr{M}_+(\mathscr{L}\cdot\mathscr{L})$ the set of all solutions of the moment problem on $\mathscr{L}\cdot\mathscr{L}$. A rational moment problem is said to be *determinate* if it has a unique solution, *indeterminate* if it has more than one solution.

3. Orthogonal rational functions

Let M be a given positive linear functional on $\mathscr{L}\cdot\mathscr{L}$, and assume that the rational Stieltjes moment problem on $\mathscr{L}\cdot\mathscr{L}$ is solvable.

We define the inner product $\langle\cdot,\cdot\rangle$ on $\mathscr{L}$ by

$$\langle g,h\rangle = M[g\cdot\overline{h}], \ \ g,h\in\mathscr{L}. \tag{3.1}$$

Equivalently $\langle g,h\rangle = \int_0^\infty g(t)\overline{h(t)}\mathrm{d}\mu(t)$, where μ is any measure in $\mathscr{M}_+(\mathscr{L}\cdot\mathscr{L})$. We can construct an *orthonormal sequence* $\{\varphi_n\}_{n=1}^\infty$ in $\mathscr{L}$ such that $\varphi_0\in\mathscr{L}_0$, $\varphi_n\in\mathscr{L}_n\setminus\mathscr{L}_{n-1}$ for $n = 1,2,\ldots$ (for example by Gram-Schmidt orthonormalization of the basis $\{\Omega_n\}$ in [8, Section 13]). Note that φ_n is unique up to a unimodular constant factor.

We may write

$$\varphi_n(z) = \frac{P_n(z)}{D_n(z)}, \ \text{ where } \ P_n\in\Pi_n\setminus\Pi_{n-1}. \tag{3.2}$$

All the zeros of the polynomial P_n are simple and contained in $(0,\infty)$. We may then normalize φ_n such that $P_n(x) < 0$ for $x\in(-\infty,0)$.

The functions σ_n *of the second kind* are defined by

$$\sigma_n(z) = M_t\left[\frac{\varphi_n(t)-\varphi_n(z)}{t-z}\right], \ \text{ for } \ n = 0,1,2,\ldots. \tag{3.3}$$

(Note that $\varphi(t) = \frac{\varphi_n(t)-\varphi_n(z)}{t-z}$ belongs to $\mathscr{L}_n$ and that M_t means that the linear functional M operates on the variable t). Thus

$$\sigma_n(z) = \int_0^\infty \frac{\varphi_n(t)-\varphi_n(z)}{t-z}\mathrm{d}\mu(t) \tag{3.4}$$

for any $\mu\in\mathscr{M}_+(\mathscr{L})$.

The functions σ_n, φ_n satisfy a three-term recurrence relation of the form

$$\begin{bmatrix}\sigma_0 & \sigma_{-1}\\ \varphi_0 & \varphi_{-1}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \begin{bmatrix}\sigma_n(z)\\ \varphi_n(z)\end{bmatrix} = b_n(z)\begin{bmatrix}\sigma_{n-1}(z)\\ \varphi_{n-1}(z)\end{bmatrix} + a_n(z)\begin{bmatrix}\sigma_{n-2}(z)\\ \varphi_{n-2}(z)\end{bmatrix}, \ \ n = 1,2,\ldots \tag{3.5}$$

where

$$a_1(z)=\frac{W_1}{r_1(z)},\ b_1(z)=\frac{Q_1z+R_1}{r_1(z)},\ a_n(z)=\frac{W_nr_{n-2}(z)}{r_n(z)},\ b_n(z)=\frac{Q_nr_{n-2}(z)+R_nr_{n-1}(z)}{r_n(z)},\ n=2,3,\dots \tag{3.6}$$

Thus

$$\begin{aligned}
a_1(z)&=\frac{W_1}{\alpha_1-z}, & b_1(z)&=\frac{Q_1z+R_1}{\alpha_1-z},\\
a_2(z)&=\frac{W_2}{\beta_1-z}, & b_2(z)&=\frac{Q_2+R_2(\alpha_1-z)}{\beta_1-z},\\
a_{2m}(z)&=\frac{W_{2m}(\beta_{m-1}-z)}{\beta_m-z}, & b_{2m}(z)&=\frac{Q_{2m}(\beta_{m-1}-z)+R_{2m}(\alpha_m-z)}{\beta_m-z},\ m=2,3,\dots\\
a_{2m+1}(z)&=\frac{W_{2m+1}(\alpha_m-z)}{\alpha_{m+1}-z}, & b_{2m+1}(z)&=\frac{Q_{2m+1}(\alpha_m-z)+R_{2m+1}(\beta_m-z)}{\alpha_{m+1}-z},\ m=1,2,\dots
\end{aligned} \tag{3.7}$$

Here Q_n,R_n,W_n are constants. (Note that Q_1 and R_1 have different meaning from that in [8], these coefficients corresponding to U_1,V_1 in [8].)

The signs of the coefficients are as follows (see [8, Section 9]):

$$\begin{aligned}
&Q_{2m}<0,\ \ Q_{2m+1}>0,\ \ \text{for all } m,\\
&R_{2m}>0,\ \ R_{2m+1}<0,\ \ \text{for all } m,\\
&W_n<0,\ \ \text{for all } n.
\end{aligned} \tag{3.8}$$

(Note that the result is obvious for R_1 and Q_1 from our normalization which assumes that the numerator of $\varphi_n=P_n/D_n$ satisfies $P_n(x)<0$ if $x\in(-\infty,0)$.)

It follows from (3.8) that $b_{2m}<0$, $b_{2m+1}>0$ for $x\in(\beta,\alpha)$.

The *determinant formula* for the orthonormal functions has the form

$$\sigma_n(z)\varphi_{n-1}(z)-\sigma_{n-1}(z)\varphi_n(z)=(-1)^{n-1}a_1(z)a_2(z)\cdots a_n(z). \tag{3.9}$$

This means

$$\begin{aligned}
\sigma_{2m}(z)\varphi_{2m-1}(z)-\sigma_{2m-1}(z)\varphi_{2m}(z)&=-\frac{W_1W_2\cdots W_{2m}}{(\beta_m-z)(\alpha_m-z)},\\
\sigma_{2m+1}(z)\varphi_{2m}(z)-\sigma_{2m}(z)\varphi_{2m+1}(z)&=\frac{W_1W_2\cdots W_{2m+1}}{(\beta_m-z)(\alpha_{m+1}-z)}.
\end{aligned} \tag{3.10}$$

The *confluent Christoffel-Darboux formula* has the form

$$r_n(z)\varphi_n(z)[r_{n-1}(z)\varphi_{n-1}(z)]'-[r_n(z)\varphi_n(z)]'r_{n-1}(z)\varphi_{n-1}(z)=(-1)^{n-1}W_1W_2\cdots W_n\sum_{k=0}^{n-1}\varphi_k(z)^2. \tag{3.11}$$

This means

$$
\begin{aligned}
&(\beta_m - z)\varphi_{2m}(z)[(\alpha_m - z)\varphi_{2m-1}(z)]' - [(\beta_m - z)\varphi_{2m}(z)]'(\alpha_m - z)\varphi_{2m-1}(z) \\
&\qquad\qquad\qquad\qquad\qquad\qquad = -W_1 W_2 \cdots W_{2m} \sum_{k=0}^{2m-1} \varphi_k(z)^2, \\
&(\alpha_{m+1} - z)\varphi_{2m+1}(z)[(\beta_m - z)\varphi_{2m}(z)]' - [(\alpha_{m+1} - z)\varphi_{2m+1}(z)]'(\beta_m - z)\varphi_{2m}(z) \\
&\qquad\qquad\qquad\qquad\qquad\qquad = W_1 W_2 \cdots W_{2m+1} \sum_{k=0}^{2m} \varphi_k(z)^2.
\end{aligned}
\tag{3.12}
$$

For all these formulas, see [8].

4. Continued fractions

For general concepts and results concerning continued fractions we refer to [12], [16].

The recurrence relation (3.5) defines a continued fraction $\mathbf{K}_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right)$, with approximants $\frac{\sigma_n}{\varphi_n}$. (When convenient, we omit the argument z in formulas in the following.) The *odd contraction* $\mathbf{K}_{m=1}^{\infty}\left(\frac{u_m}{v_m}\right)$ of $\mathbf{K}_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right)$ has approximants $\frac{\sigma_{2m+1}}{\varphi_{2m+1}}$ and satisfies the recurrence relation

$$
\begin{bmatrix} \sigma_{2m+1} \\ \varphi_{2m+1} \end{bmatrix} = v_m \begin{bmatrix} \sigma_{2m-1} \\ \varphi_{2m-1} \end{bmatrix} + u_m \begin{bmatrix} \sigma_{2m-3} \\ \varphi_{2m-3} \end{bmatrix},
\tag{4.1}
$$

where

$$
u_m = -\frac{a_{2m-1}a_{2m}b_{2m+1}}{b_{2m-1}}, \quad v_m = \frac{b_{2m-1}b_{2m}b_{2m+1} + a_{2m}b_{2m+1} + a_{2m+1}b_{2m-1}}{b_{2m-1}}
\tag{4.2}
$$

for $m = 1, 2, \ldots$. See e.g., [12, Section 2.2.4].

In particular, this gives in our situation

$$
u_m = -\frac{(\alpha_{m-1} - z)(\beta_{m-1} - z)b_{2m+1}(z)}{(\alpha_m - z)(\beta_m - z)b_{2m-1}(z)}.
\tag{4.3}
$$

We shall make use of these formulas in Section 8. Note that for each m, $b_{2m+1}(z)$ has at most one zero, which is real. The odd contraction is defined only for those z which are not a zero for any $b_{2m+1}(z)$.

A continued fraction $\mathbf{K}_{n=1}^{\infty}\left(\frac{c_n}{d_n}\right)$ is said to be *equivalent* to the continued fraction $\mathbf{K}_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right)$ if the nth approximant of $\mathbf{K}_{n=1}^{\infty}\left(\frac{c_n}{d_n}\right)$ equals the nth approximant of $\mathbf{K}_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right)$. It will be convenient in the development of our results to make use of a continued fraction of the form $\mathbf{K}_{n=1}^{\infty}\left(\frac{1}{d_n}\right)$, equivalent to $\mathbf{K}_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right)$. The elements (recurrence coefficients) $d_n(z)$ have the form

$$
d_n = \rho_n b_n,
\tag{4.4}
$$

where

$$\rho_1 = 1,\ \ \rho_{2m+1} = \frac{a_2a_4\cdots a_{2m}}{a_1a_3\cdots a_{2m+1}},\ \ \rho_{2m} = \frac{a_1a_3\cdots a_{2m-1}}{a_2a_4\cdots a_{2m}},\ \ m = 1,2,\dots. \tag{4.5}$$

See e.g., [12, Section 2.2.2]. We shall denote the canonical numerators and denominators of $\mathbf{K}_{n=1}^{\infty}\left(\frac{1}{d_n}\right)$ as $\Sigma_n(z)$ and $\Phi_n(z)$. Then

$$\Phi_n = \rho_1\rho_2\cdots\rho_n\varphi_n,\ \ \Sigma_n = \rho_1\rho_2\cdots\rho_n\sigma_n. \tag{4.6}$$

In our case this gives

$$d_{2m} = \frac{W_1W_3\cdots W_{2m-1}r_{2m}}{W_2W_4\cdots W_{2m}r_{2m-1}}b_{2m},\quad d_{2m+1} = \frac{W_2W_4\cdots W_{2m}r_{2m+1}}{W_1W_3\cdots W_{2m+1}r_{2m}}b_{2m+1}. \tag{4.7}$$

Written out this formula gives

$$\begin{aligned} d_{2m} &= \frac{W_1W_3\cdots W_{2m-1}}{W_2W_4\cdots W_{2m}}\cdot\frac{Q_{2m}(\beta_{m-1}-z)+R_{2m}(\alpha_m-z)}{\alpha_m-z},\\ d_{2m+1} &= \frac{W_2W_4\cdots W_{2m}}{W_1W_3\cdots W_{2m+1}}\cdot\frac{Q_{2m+1}(\alpha_m-z)+R_{2m+1}(\beta_m-z)}{\beta_m-z}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \Phi_{2m}(z) &= \frac{\beta_m-z}{W_2W_4\cdots W_{2m}}\varphi_{2m}(z), & \Sigma_{2m}(z) &= \frac{\beta_m-z}{W_2W_4\cdots W_{2m}}\sigma_{2m}(z)\\ \Phi_{2m+1}(z) &= \frac{\alpha_{m+1}-z}{W_1W_3\cdots W_{2m+1}}\varphi_{2m+1}(z), & \Sigma_{2m+1}(z) &= \frac{\alpha_{m+1}-z}{W_1W_3\cdots W_{2m+1}}\sigma_{2m+1}(z). \end{aligned} \tag{4.9}$$

Note that Φ_n and φ_n have the same zeros, and Σ_n and σ_n have the same zeros, since $\beta_m - z$ is a factor in $D_{2m}(z)$ and $\alpha_{m+1}-z$ is a factor in $D_{2m+1}(z)$.

From the established relationships between φ_n and Φ_n and σ_n and Σ_n, the determinant formula (3.10) takes the form

$$\begin{aligned} \Sigma_{2m}\Phi_{2m-1} - \Sigma_{2m-1}\Phi_{2m} &= -1\\ \Sigma_{2m+1}\Phi_{2m} - \Sigma_{2m}\Phi_{2m+1} &= 1, \end{aligned} \tag{4.10}$$

while the confluent Christoffel-Darboux formula (3.12) takes the form

$$\begin{aligned} \Phi_{2m}\Phi'_{2m-1} - \Phi'_{2m}\Phi_{2m-1} &= -\sum_{k=0}^{2m-1}\varphi_k^2\\ \Phi_{2m+1}\Phi'_{2m} - \Phi'_{2m+1}\Phi_{2m} &= \sum_{k=0}^{2m}\varphi_k^2. \end{aligned} \tag{4.11}$$

We state a few crucial facts as propositions.

Proposition 4.1. *Let $x \in (\alpha,\beta)$. Then $d_n(x) > 0$ for all n.*

PROOF. This result follows from (3.8), (4.8) and the assumption $\alpha_k \geq \alpha$, $\beta_k \leq \beta$. ■

Proposition 4.2. *The following pairs of functions have no common zero:*

(i) Φ_n *and* Φ_{n-1}
(ii) Σ_n *and* Σ_{n-1}
(iii) Φ_n *and* Σ_n.

PROOF. This follows immediately form (4.10). ∎

Proposition 4.3. *Between two consecutive zeros of* Φ_{n+1}*, there is exactly one zero of* Φ_n.

PROOF. This result follows by a standard argument form (4.11), where we take into account the intermediate value theorem for continuous functions, the facts that the zeros of all Φ_n are simple and contained in $(0,\infty)$, and the fact that $\sum_{k=0}^{\infty} \varphi_k(x)^2$ is always positive for real x. ∎

5. Natural solutions

For all the results in this section we refer especially to [8, Sections 12-13]. See also [6], [7].

Let $x_{n,1}, x_{n,2}, \ldots, x_{n,n}$ be the zeros of φ_n. Then there exist positive weights $\lambda_{n,1}, \lambda_{n,1}, \ldots, \lambda_{n,n}$ such that

$$\int_0^\infty f(t)\mathrm{d}\mu(t) = \sum_{k=1}^n \lambda_{n,k} f(x_{n,k}) \tag{5.1}$$

for all $f \in \mathscr{L}_n \cdot \mathscr{L}_{n-1}$, $\mu \in \mathscr{M}_+(\mathscr{L} \cdot \mathscr{L})$. For a given z, the function $\varphi(t) = \frac{\varphi_n(t)-\varphi_n(z)}{t-z}$ belongs to $\mathscr{L}_n$ and therefore to $\mathscr{L}_n \cdot \mathscr{L}_{n-1}$. Hence the above quadrature formula implies that $\sigma_n(z) = \sum_{k=1}^n \lambda_{n,k} \frac{\varphi_n(x_{n,k})-\varphi_n(z)}{x_{n,k}-z}$ (recall the definition (3.3)-(3.4)), and thus

$$\sigma_n(z) = \varphi_n(z) \sum_{k=1}^n \frac{\lambda_{n,k}}{z - x_{n,k}}, \tag{5.2}$$

since $\varphi_n(x_{n,k}) = 0$. Let μ_n be the discrete measure with mass of size $\lambda_{n,k}$ at the point $x_{n,k}$ for $k = 1, 2, \ldots, n$. Then we may write (5.2) as

$$\frac{\Sigma_n(z)}{\Phi_n(z)} = \frac{\sigma_n(z)}{\varphi_n(z)} = \int_0^\infty \frac{\mathrm{d}\mu_n(t)}{z-t} = S(z, \mu_n). \tag{5.3}$$

By using the determinant formula (3.10) and the formula (see e.g., [12, (1.2.10)])

$$\frac{\sigma_n(z)}{\varphi_n(z)} - \frac{\sigma_{n-2}(z)}{\varphi_{n-1}(z)} = b_n(z) \frac{\varphi_{n-1}(z)}{\varphi_n(z)} \left[\frac{\sigma_{n-1}(z)}{\varphi_{n-1}(z)} - \frac{\sigma_{n-2}(z)}{\varphi_{n-2}(z)} \right] \tag{5.4}$$

it can be shown that the sequence $\left\{\frac{\Sigma_{2m}(x)}{\Phi_{2m}(x)}\right\} = \left\{\frac{\sigma_{2m}(x)}{\varphi_{2m}(x)}\right\}$ decreases on (β, α), and the sequence $\left\{\frac{\Sigma_{2m+1}(x)}{\Phi_{2m+1}(x)}\right\} = \left\{\frac{\sigma_{2m+1}(x)}{\varphi_{2m+1}(x)}\right\}$ increases on (β, α). Furthermore it follows from (3.10) that $\frac{\sigma_{2p}(x)}{\varphi_{2p}(x)} > \frac{\sigma_{2q+1}(x)}{\varphi_{2q+1}(x)}$ for any p, q and $x \in (\beta, \alpha)$. In particular $\left\{\frac{\sigma_{2m}(x)}{\varphi_{2m}(x)}\right\}$ is bounded below on (β, α) and $\left\{\frac{\sigma_{2m+1}(x)}{\varphi_{2m+1}(x)}\right\}$ is bounded above on (β, α). Consequently the two sequences converge on (β, α). By an argument using properties of normal families, taking into account the quadrature formula

(5.2), it follows that $\left\{\frac{\sigma_{2m}(z)}{\varphi_{2m}(z)}\right\}$ and $\left\{\frac{\sigma_{2m+1}(z)}{\varphi_{2m+1}(z)}\right\}$ converge locally uniformly in $\mathbb{C}\setminus[0,\infty)$ to holomorphic functions $F_0(z)$ and $F_\infty(z)$. By a standard compactness argument applied to the sequence of measures $\{\mu_n\}$, it follows that the functions F_0 and F_∞ are Stieltjes transforms of measures μ_0 and μ_∞ with support in $[0,\infty)$ representing the functional M on $\mathscr{L}$. I.e.,

$$F_0(z) = S(z,\mu_0), \quad F_\infty(z) = S(z,\mu_\infty) \tag{5.5}$$

where $\mu_0,\mu_\infty \in \mathscr{M}_+(\mathscr{L})$.

The measures μ_0 and μ_∞ are called *natural solutions* of the moment problem on $\mathscr{L}$.

Our aim is to investigate structure and properties of the support of μ_0 and μ_∞. We state as a proposition a preliminary result in this direction.

We define

$$Z_n = \text{ The set of all zeros of } \varphi_n. \tag{5.6}$$

Proposition 5.1. *Every point in* supp μ_0 *(resp.* supp μ_∞*) is an accumulation point for sequences in* $\cup_{m=1}^{\infty} Z_{2m}$ *(resp.* $\cup_{m=0}^{\infty} Z_{2m+1}$*).*

PROOF. This is a direct consequence of the construction of μ_0 and μ_∞ through weak star convergence of $\{\mu_{2m}\}$ and $\{\mu_{2m+1}\}$. ■

We also state as a proposition a fact that will be crucial in the discussion in Section 6.

Proposition 5.2. *For any* $\mu \in \mathscr{M}_+(\mathscr{L}\cdot\mathscr{L})$ *we have*

$$F_\infty(z) = S(x,\mu_\infty) \le S(x,\mu) \le S(x,\mu_0) = F_0(x) \tag{5.7}$$

for $x \in (\beta,\alpha)$.

PROOF. The result follows from [8, Theorem 13.2]. ■

The functions $F_0(z) = S(z,\mu_0)$ and $F_\infty(z) = S(z,\mu_\infty)$ map the open upper half-plane into the open lower half-plane (none of the functions being a constant). Consequently the singularities of F_0 and F_∞ in $(0,\infty)$ are simple poles in $\mathbb{R}$.

6. Indeterminate problems

We shall in this section consider a rational Stieltjes moment problem on $\mathscr{L}\cdot\mathscr{L}$ which is indeterminate, i.e., $\mathscr{M}_+(\mathscr{L}\cdot\mathscr{L})$ contains more than one element. We shall make essential use of the continued fraction $\mathbf{K}_{n=1}^{\infty}\left(\frac{1}{d_n}\right)$ introduced in Section 4. Recall that the even and odd approximants $\frac{\Sigma_{2m}(z)}{\Phi_{2m}(z)}$ and $\frac{\Sigma_{2m+1}(z)}{\Phi_{2m+1}(z)}$ converge to the Stieltjes transforms $S(z,\mu_0)$ and $S(z,\mu_\infty)$.

Proposition 6.1. *For* $x \in (\beta,\alpha)$ *the series* $\sum_{n=1}^{\infty} d_n(x)$ *converges.*

PROOF. Since the moment problem on $\mathscr{L}\cdot\mathscr{L}$ in indeterminate, it follows from Proposition 5.2 that the functions $S(x,\mu_0)$ and $S(x,\mu_\infty)$ do not coincide for $x\in(\beta,\alpha)$. This implies that the sequence $\left\{\frac{\Sigma_n(x)}{\Phi_n(x)}\right\}$ does not converge. In other words, the continued fraction $\mathbf{K}_{n=1}^{\infty}\left(\frac{1}{d_n(x)}\right)$ does not converge. Since $d_n(x)>0$ for $x\in(\beta,\alpha)$ by Proposition 4.1, the series $\sum_{n=1}^{\infty} d_n(x)$ converges according to *Van Vleck's criterion*. see e.g., [12, p. 142]. ∎

It follows from (4.8) and Proposition 6.1 that for $x\in(\beta,\alpha)$ we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\frac{W_1W_3\cdots W_{2m-1}}{W_2W_4\cdots W_{2m}}\left[Q_{2m}\frac{\beta_{m-1}-x}{\alpha_m-x}+R_{2m}\right]<\infty,\\
&\sum_{m=1}^{\infty}\frac{W_2W_4\cdots W_{2m}}{W_1W_3\cdots W_{2m+1}}\left[Q_{2m+1}\frac{\alpha_m-x}{\beta_m-x}+R_{2m+1}\right]<\infty.
\end{aligned}\tag{6.1}$$

We wish to establish convergence of the sequence $\{\Phi_{2m+1}(z)\}$, $\{\Sigma_{2m+1}(z)\}$, $\{\Phi_{2m}(z)\}$, and $\{\Sigma_{2m}(z)\}$. We shall show that this holds true for a indeterminate problem for z in compact subsets of $\mathbb{C}\setminus S$ with S compact. This will be a generalization of the treatment of the two-point situation (see [1], [2], [3]).

We shall use the following simple observation. Let K be a compact subset of $\mathbb{C}\setminus S$ and define

$$\delta=\inf\{|z-\gamma|:z\in K,\gamma\in S\}\quad\text{and}\quad\Delta=\sup\{|z-\gamma|:z\in K,\gamma\in S\}.\tag{6.2}$$

then it is clear that for $z\in K$

$$\frac{\delta}{\Delta}\le\left|\frac{\beta_{m-1}-z}{\alpha_m-z}\right|\le\frac{\Delta}{\delta}\quad\text{and}\quad\frac{\delta}{\Delta}\le\left|\frac{\alpha_m-z}{\beta_m-z}\right|\le\frac{\Delta}{\delta}.\tag{6.3}$$

The following proposition will be an essential tool.

Proposition 6.2. *Suppose that all β_k are in a compact subset of $(-\infty,\beta]$, then the rational Stieltjes moment problem on $\mathscr{L}\cdot\mathscr{L}$ is indeterminate if and only if there exist finite numbers Γ_o, Γ_e, Λ_o and Λ_e such that*

$$\sum_{m=1}^{\infty}\left|R_{2m}\frac{W_1W_3\cdots W_{2m-1}}{W_2W_4\cdots W_{2m}}\right|\le\Gamma_e<\infty,\quad\sum_{m=1}^{\infty}\left|R_{2m+1}\frac{W_2W_4\cdots W_{2m}}{W_1W_3\cdots W_{2m+1}}\right|\le\Gamma_o<\infty,\tag{6.4}$$

and

$$\sum_{m=1}^{\infty}\left|Q_{2m}\frac{W_1W_3\cdots W_{2m-1}}{W_2W_4\cdots W_{2m}}\right|\le\Lambda_e<\infty,\quad\sum_{m=1}^{\infty}\left|Q_{2m+1}\frac{W_2W_4\cdots W_{2m}}{W_1W_3\cdots W_{2m+1}}\right|\le\Lambda_o<\infty.\tag{6.5}$$

PROOF. For $x\in(\beta,\alpha)$ it holds by (3.8) that

$$\left|Q_{2m}\frac{\beta_{m-1}-x}{\alpha_m-x}+R_{2m}\right|\ge|R_{2m}|\ \text{ and }\ \left|Q_{2m+1}\frac{\alpha_m-x}{\beta_m-x}+R_{2m+1}\right|\ge|R_{2m+1}|.$$

So that for an indeterminate problem, (6.1) immediately implies

$$\sum_{m=1}^{\infty}\left|R_{2m}\frac{W_1W_3\cdots W_{2m-1}}{W_2W_4\cdots W_{2m}}\right|\leq\Lambda_e<\infty \text{ and } \sum_{m=1}^{\infty}\left|R_{2m+1}\frac{W_2W_4\cdots W_{2m}}{W_1W_3\cdots W_{2m+1}}\right|\leq\Lambda_o<\infty.$$

From (6.1) and (6.3), it follows that for $z\in K\cap(\beta,\alpha)$

$$\sum_{m=1}^{\infty}\left|Q_{2m}\frac{W_1W_3\cdots W_{2m-1}}{W_2W_4\cdots W_{2m}}\frac{\delta}{\Delta}\right|\leq\sum_{m=1}^{\infty}\left|Q_{2m}\frac{W_1W_3\cdots W_{2m-1}}{W_2W_4\cdots W_{2m}}\right|\left|\frac{\beta_{m-1}-z}{\alpha_m-z}\right|<\infty$$

This means that there exists a finite Γ_e such that

$$\sum_{m=1}^{\infty}\left|Q_{2m}\frac{W_1W_3\cdots W_{2m-1}}{W_2W_4\cdots W_{2m}}\right|\leq\Gamma_e<\infty,$$

and similarly, there is a finite constant Γ_o such that

$$\sum_{m=1}^{\infty}\left|Q_{2m+1}\frac{W_2W_4\cdots W_{2m}}{W_1W_3\cdots W_{2m+1}}\right|\leq\Gamma_o<\infty.$$

This concludes the proof in one direction.

In the opposite direction we should prove that (6.4) and (6.5) imply the indeterminacy of the rational Stieltjes moment problem on $\mathscr{L}\cdot\mathscr{L}$. Using (6.3), (6.4) and (6.5) it is easy to show that for $z\in K$

$$\sum_{m=1}^{\infty}d_{2m}(z)\leq\sum_{m=1}^{\infty}R_{2m}\frac{W_1W_3\cdots W_{2m-1}}{W_2W_4\cdots W_{2m}}+\sum_{m=1}^{\infty}\left|Q_{2m}\frac{W_1W_3\cdots W_{2m-1}}{W_2W_4\cdots W_{2m}}\right|\frac{\Delta}{\delta}\leq\Lambda_e+\Gamma_e\frac{\Delta}{\delta}=D_e<\infty.$$

Similarly

$$\sum_{m=1}^{\infty}d_{2m+1}(z)\leq\Lambda_o+\Gamma_o\frac{\Delta}{\delta}=D_o<\infty.$$

Hence also $\sum_n d_n(z)<D_e+D_o$ will converge for $z\in K$. Thus $\mathbf{K}\left(\frac{1}{d_n(z)}\right)$ diverges [12, p. 142] while we know that the two sequences $\left\{\frac{\Sigma_{2m+1}}{\Phi_{2m+1}}\right\}$ and $\left\{\frac{\Sigma_{2m}}{\Phi_{2m}}\right\}$ converge on subsets of $\mathbb{C}\setminus[0,\infty)$ to F_∞ and F_0 respectively. Since the continued fraction diverges, these two limiting functions must be distinc, which means that the moment problem is indeterminate. ■

The proof of the next theorem now follows closely the arguments used in [4, Theorem 2.8].

Theorem 6.3. *Let the rational Stieltjes moment problem on $\mathscr{L}\cdot\mathscr{L}$ be indeterminate and assume that the set of poles is contained in a compact set $S=\mathrm{cl}(A)\cup\mathrm{cl}(B)$. Then the sequences $\{\Phi_{2m}(z)\}$, $\{\Phi_{2m+1}(z)\}$, $\{\Sigma_{2m}(z)\}$ and $\{\Sigma_{2m+1}(z)\}$ converge locally uniformly in $\mathbb{C}\setminus S$ to holomorphic functions $\Phi_0(z)$, $\Phi_\infty(z)$, $\Sigma_0(z)$ and $\Sigma_\infty(z)$ and it holds that*

$$\Sigma_\infty(z)\Phi_0(z)-\Sigma_0(z)\Phi_\infty(z)=1, \tag{6.6}$$

and

$$\Phi_\infty(z)\Phi_0'(z)-\Phi_\infty'(z)\Phi_0(z)=\sum_{k=0}^{\infty}\varphi_k(z)^2. \tag{6.7}$$

PROOF. Let K be a compact set in $\mathbb{C}\setminus S$, then, according to Proposition 6.2, we have $\sum_{n=1}^{\infty} d_n(z) \leq D_e + D_o = D < \infty$ for $z \in K$.
Next we use the recurrence relations

$$\begin{bmatrix} \Sigma_{-1}(z) & \Sigma_0(z) \\ \Phi_{-1}(z) & \Phi_0(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \Sigma_n(z) \\ \Phi_n(z) \end{bmatrix} = d_n(z) \begin{bmatrix} \Sigma_{n-1}(z) \\ \Phi_{n-1}(z) \end{bmatrix} + \begin{bmatrix} \Sigma_{n-2}(z) \\ \Phi_{n-2}(z) \end{bmatrix}, \quad n \geq 1,$$

which yield

$$|\Phi_n(z)| \leq \prod_{k=1}^{n} (1 + d_n(z)) \quad \text{and} \quad |\Sigma_n(z)| \leq \prod_{k=1}^{n} (1 + d_n(z)).$$

Taking the logarithm, and using $\log(1+d) < d$ for $d > 0$ we get from this that

$$|\Phi_n(z)| \text{ and } |\Sigma_n(z)| \text{ are bounded by } \exp\left\{\sum_{k=1}^{n} d_n(z)\right\} = \mathrm{e}^D.$$

Thus both

$$\sum_{n=1}^{\infty} |d_n \Sigma_n(z)| \quad \text{and} \quad \sum_{n=1}^{\infty} |d_n \Phi_n(z)| \text{ are bounded by } D\mathrm{e}^D.$$

Hence $\sum_{n=1}^{\infty} d_n \Sigma_n(z)$ and $\sum_{n=1}^{\infty} |d_n \Phi_n(z)|$ converge uniformly and absolutely on K.

Using the recurrence relation again, we deduce

$$\Sigma_{2m}(z) = \sum_{k=1}^{m} d_{2k}(z)\Sigma_{2k-1}(z), \;\; \Phi_{2m}(z) = 1 + \sum_{k=1}^{m} d_{2k}(z)\Phi_{2k-1}(z), \;\; m = 1, 2, \ldots$$

and

$$\Sigma_{2m+1}(z) = 1 + \sum_{k=1}^{m} d_{2k+1}(z)\Sigma_{2k}(z), \;\; \Phi_{2m+1}(z) = \sum_{k=1}^{m} d_{2k+1}(z)\Phi_{2k}(z), \;\; m = 1, 2, \ldots$$

so that all the sequences $\{\Sigma_{2m}(z)\}_{m=1}^{\infty}$, $\{\Sigma_{2m+1}(z)\}_{m=1}^{\infty}$, $\{\Phi_{2m}(z)\}_{m=1}^{\infty}$, $\{\Phi_{2m+1}(z)\}_{m=1}^{\infty}$ converge uniformly on K.
The equalities (6.6) and (6.7) follow from the determinant relation (4.10) and the confluent Christoffel-Darboux formula (4.11) by letting m tend to ∞. ■

7. Support of natural solutions

We continue to assume the conditions and hence also the results of the foregoing section. We then have

$$F_0(z) = S(z, \mu_0) = \frac{\Sigma_0(z)}{\Phi_0(z)}, \;\; F_\infty(z) = S(z, \mu_\infty) = \frac{\Sigma_\infty(z)}{\Phi_\infty(z)}. \tag{7.1}$$

Consequently F_0 and F_∞ are meromorphic functions in $\mathbb{C}\setminus S$. We have established in Section 5 that the singularities of F_0 and F_∞ in $(0,\infty)$ are simple poles. Since the support of μ_0 and of μ_∞ are contained in $[0,\infty)$, it follows that supp μ_0 consists of the poles of F_0 in $(0,\infty)$ and possibly the origin, while supp μ_∞ consists of the poles of F_∞ in $(0,\infty)$ and possibly the origin.

From (6.6) we conclude that each of the pairs (Φ_0,Φ_∞), (Σ_0,Σ_∞), (Φ_0,Σ_0) and $(\Phi_\infty,\Sigma_\infty)$ have no common zero. Since the poles of F_0 and F_∞ are simple, this implies that the zeros of Φ_0 and Φ_∞ are simple zeros. Arguing as in Section 4 we conclude from (6.7) that between consecutive zeros of Φ_0 there is a zero of Φ_∞ and *vice versa*.

For the sake of clarity we state the following result as a proposition.

Proposition 7.1. $\operatorname{supp}\mu_0\cap(0,\infty)$ *consists of all zeros of* Φ_0 *and* $\operatorname{supp}\mu_\infty\cap(0,\infty)$ *consists of all zeros of* Φ_∞.

PROOF. This follows immediately from the fact that $\operatorname{supp}\mu_0\cap(0,\infty)$ (resp. $\operatorname{supp}\mu_\infty\cap(0,\infty)$) consists of all poles in $(0,\infty)$ of F_0 (resp. F_∞), and these poles are exactly the zeros of Φ_0 (resp. Φ_∞). ■

We shall henceforth denote the zeros of Φ_{2m} by $x_k^{(2m)}$, $k=-m,\dots,-1,1,\dots,m$, and the zeros of Φ_{2m+1} by $x_k^{(2m+1)}$, $k=-m,\dots,-1,0,1,\dots,m$. They are ordered by size in the following sense:

$$\begin{gathered}x_{-m}^{(2m)}<x_{-m+1}^{(2m)}<\cdots<x_{-1}^{(2m)}<x_{1}^{(2m)}<\cdots<x_{m-1}^{(2m)}<x_{m}^{(2m)},\quad\text{and}\\ x_{-m}^{(2m+1)}<x_{-m+1}^{(2m+1)}<\cdots<x_{-1}^{(2m+1)}<x_{0}^{(2m+1)}<x_{1}^{(2m+1)}<\cdots<x_{m-1}^{(2m+1)}<x_{m}^{(2m+1)}.\end{gathered}\tag{7.2}$$

Note that the smallest and the largest zero of Φ_n can be written as $x^{(n)}_{-[\frac{n}{2}]}$ and $x^{(n)}_{[\frac{n}{2}]}$. From the separation properties of the zeros it follows that for each $k=0,1,2,\dots$, the sequence $\{x^{(n)}_{-[\frac{n}{2}]+k}\}_{n\ge 2k+1}$ decreases to a values $\xi_k\in[0,\infty)$ and the sequence $\{x^{(n)}_{[\frac{n}{2}]-k}\}_{n\ge 2k+1}$ increases to a value $\eta_k\in(0,\infty]$.

We shall now concentrate on the central zero of Φ_{2m+1}. For Φ_1, this central zero is clearly

$$x_0=-\frac{R_1}{Q_1}\tag{7.3}$$

since there is only one. It follows from (3.8) that $x_0\in(0,\infty)$. Furthermore we make the crucial assumption that

$$\frac{\alpha_m Q_{2m+1}+\beta_m R_{2m+1}}{Q_{2m+1}+R_{2m+1}}=x_0\ \text{ for all } m.\tag{7.4}$$

Observe that the condition (7.4) is equivalent to $b_{2m+1}(x_0)=0$ *for all m where the* b_{2m+1} *are as defined in (3.7).*

This assumption will fix x_0 as being the central zero of all subsequent Φ_{2m+1} as shown below.

Proposition 7.2. *The middle zero of* Φ_{2m+1} *equals* x_0*, i.e.,*

$$x_0^{(2m+1)}=x_0\ \text{ for }\ m=0,1,2,\dots.\tag{7.5}$$

PROOF. From (3.5) and (3.7) we find that $\varphi_1(z)=\frac{Q_1z+R_1}{\alpha_1-z}$. By the definition of x_0 (formula (7.3)) we then get $\varphi_1(x_0)=0$. Since the condition (7.4) is equivalent to $b_{2m+1}(x_0)=0$ for all m, we conclude by induction, using (3.5) and (3.7), that $\varphi_{2m+1}(x_0)=0$, hence $\Phi_{2m+1}(x_0)=0$ for all m.

By separation properties of zeros we have $x_{-1}^{(2)} < x_0^{(1)} < x_1^{(2)}$ and $x_0^{(1)} = x_0$ since x_0 is the only zero of Q_1. It follows that $x_{-1}^{(3)} < x_1^{(2)} < x_0 < x_1^{(2)} < x_1^{(3)}$, which means that $x_0^{(3)} = x_0$. Arguing step by step in this way, we find that $x_0^{(2m+1)} = x_0$ for all m. ■

It follows from Proposition 7.2 that x_0 is a zero of Φ_∞. We denote the zeros of Φ_0 by $x_p^{(0)}$, $p = \pm 1, \pm 2, \ldots$, and the zeros of Φ_∞ by $x_p^{(\infty)}$, $p = 0, \pm 1, \pm 2, \ldots$, with $x_0^{(\infty)} = x_0$, ordered such that

$$\cdots < x_{-2}^{(0)} < x_{-1}^{(\infty)} < x_{-1}^{(0)} < x_0^{(\infty)} < x_1^{(0)} < x_1^{(\infty)} < x_2^{(0)} < \cdots \tag{7.6}$$

(recall that the zeros of Φ_0 and Φ_∞ separate each other). I.e.,

$$x_{-p}^{(0)} < x_{-p+1}^{(\infty)} < x_{-p+1}^{(0)} \quad \text{and} \quad x_p^{(0)} < x_p^{(\infty)} < x_{p+1}^{(0)} \quad \text{for } p = 2, 3, \ldots, \text{ and} \quad x_{-1}^{(0)} < x_0^{(\infty)} < x_1^{(0)}. \tag{7.7}$$

(Cf. the discussion just before Proposition 7.1.)

We recall that Z_n denotes the set of zeros of Φ_n. Furthermore the support of μ_0 (resp. μ_∞) consists of the zeros of Φ_0 (resp. Φ_∞), plus possibly the origin. Except for the origin, supp μ_0 and supp μ_∞ are disjoint.

Proposition 7.3. supp $\mu_0 \cap (0, \infty)$ *consists of all accumulation points in* $(0, \infty)$ *of the set* $\cup_{m=1}^{\infty} Z_{2m}$ *and* supp $\mu_\infty \cap (0, \infty)$ *consists of all accumulation points in* $(0, \infty)$ *of the set* $\cup_{m=0}^{\infty} Z_{2m+1}$.

PROOF. We already have from Proposition 5.1 that in the general situation every point in supp $\mu_0 \cap (0, \infty)$ (resp. supp $\mu_\infty \cap (0, \infty)$) is an accumulation point of $\cup_{m=1}^{\infty} Z_{2m}$ (resp. $\cup_{m=0}^{\infty} Z_{2m+1}$). It follows from *Hurzwitz' theorem* (see e,g., [13, Part II, p.49]) that every accumulation point in $(0, \infty)$ of $\cup_{m=1}^{\infty} Z_{2m}$ (resp. $\cup_{m=0}^{\infty} Z_{2m+1}$) is a zero of Φ_0 (resp. Φ_∞), hence belongs to supp μ_0 (resp. supp μ_∞). ■

Let ξ_k be any of the limits discussed earlier in this section. It follows from Proposition 7.3 that if $\xi_k \in (0, \infty)$, then ξ_k belongs to the support of both Φ_0 and Φ_∞. This is a contradiction, and hence we conclude that $\xi_k = 0$ for all k. Similarly we find that $\eta_k = \infty$ for all k. This is in contrast to the classical situation.

Thus the sequences $\{x_{-m+k}^{(2m)}\}_m$ and $\{x_{m-k}^{(2m)}\}_m$ do not converge to support points for μ_0 in $(0, \infty)$, and the sequences $\{x_{-m+k}^{(2m+1)}\}_m$ and $\{x_{m-k}^{(2m+1)}\}_m$ do not converge to support points for μ_∞ in $(0, \infty)$.

On the other hand, according to Proposition 7.3, every point in supp $\mu_0 \cap (0, \infty)$ is an accumulation point for zeros of $\{\Phi_{2m}\}$ and every point in supp $\mu_\infty \cap (0, \infty)$ is an accumulation point for zeros of $\{\Phi_{2m+1}\}$. We may then pose the question: Can anything be said about *how* zeros of the orthogonal functions approach zeros of the limit functions? In general no such results are known, but in the two final sections we shall identify sequences of zeros of Φ_{2m} (resp. Φ_{2m+1}) which converge to each zero of Φ_0 (resp. Φ_∞) in the setting we have assumed from Section 6 on.

8. Monotonicity of zero sequences

We assume as before that the rational Stieltjes moment problem on $\mathscr{L} \cdot \mathscr{L}$ is indeterminate. We furthermore assume that (7.4) is satisfied with x_0 as in (7.3).

The arguments in this and the next section are strongly influenced by ideas from the papers [3], [2], which again have their origin in the thesis [1]. Our proofs of Theorem 8.4 and Theorem 9.1 are quite similar to the proofs of the corresponding results in the two-point situation in [3]. For the sake of completeness, we carry out the arguments.

Recall that the zeros of Φ_n coincide with zeros of P_n, where as before $\varphi_n(z) = \frac{P_n(z)}{D_n(z)}$.

Proposition 8.1.

$$\lim_{x\to\infty} P_{2m}(x) = -\infty, \quad \lim_{x\to\infty} P_{2m+1}(x) = \infty. \tag{8.1}$$

PROOF. We established in Section 3 that $P_n(x) < 0$ for $x \in (-\infty, 0)$ for all n. The limit values in (8.1) then follow from the fact that all the zeros of P_n are simple and contained in $(0,\infty)$. ∎

Recall that $Q_{2m+1} > 0$, $R_{2m+1} < 0$ (cf. (3.8)).

Proposition 8.2. *We have*

$$|R_{2m+1}| < Q_{2m+1} \tag{8.2}$$

PROOF. According to (7.4) we have

$$\frac{\alpha_m Q_{2m+1} + \beta_m R_{2m+1}}{Q_{2m+1} + R_{2m+1}} = x_0.$$

Since x_0 is finite, $|R_{2m+1}| \neq Q_{2m+1}$. If $|R_{2m+1}| > Q_{2m+1}$, then the denominator in the equation is negative while the numerator is positive (recall $|\beta_m| > |\alpha_m|$). Since $x_0 \in (0,\infty)$, this is a contradiction. ∎

Proposition 8.3. *Let $P_{2m-1}(\xi) = 0$, $\xi \neq x_0$. Then $P_{2m+1}(\xi)$ and $P_{2m-3}(\xi)$ have opposite sign.*

PROOF. It follows from (4.1–4.3) that

$$\varphi_{2m+1}(\xi) = -\frac{(\alpha_{m-1} - \xi)(\beta_{m-1} - \xi) b_{2m+1}(\xi)}{(\alpha_m - \xi)(\beta_m - \xi) b_{2m-1}(\xi)} \varphi_{2m-3}(\xi). \tag{8.3}$$

We may write

$$b_{2k+1}(z) = \frac{(Q_{2k+1} + R_{2k+1})(z - x_0)}{z - \alpha_{k+1}}.$$

Then according to Proposition 8.2 we have $b_{2k+1}(x) < 0$ for $x \in (0, x_0)$, $b_{2k+1}(x) > 0$ for $x \in (x_0, \infty)$, for all k. Hence we conclude that the signs of the factors on the right-hand side of (8.3) cancel, and the result follows. ∎

Theorem 8.4. *For every $p = 1,2,3,\ldots$, the sequence $\{x_p^{(2m+1)}\}_{m=p}^{\infty}$ is decreasing and for $p = -1,-2,-3,\ldots$ the sequence $\{x_p^{(2m+1)}\}_{m=-p}^{\infty}$ is increasing, i.e.,*

$$\begin{aligned} x_p^{(2m+1)} &< x_p^{(2m-1)}, \ \textit{for} \ \ p = 1,2,\ldots, \ \ m > p \\ x_p^{(2m+1)} &> x_p^{(2m-1)}, \ \textit{for} \ \ p = -1,-2,\ldots, \ \ m > -p. \end{aligned} \tag{8.4}$$

PROOF. We first consider $p > 0$. Recall that $x_0^{(2m+1)} = x_0$ for all m. It follows from Propositions 8.1 and 8.3 that $P_5(x_1^{(3)}) < 0$ since $P_1(x) > 0$ for $x > x_0^{(1)}$, and $x_1^{(3)} > x_0^{(1)}$. From (7.7) we have $x_0^{(3)} < x_1^{(3)} < x_2^{(5)}$. Since $\lim_{x\to\infty} P_5(x) = \infty$ by Proposition 8.3, we have for $x \in (x_0^{(5)}, x_1^{(5)})$. We then conclude from $P_5(x_1^{(3)}) < 0$ that $x_1^{(3)} \in (x_1^{(5)}, x_2^{(5)})$. Thus

$$x_1^{(5)} < x_1^{(3)}. \tag{8.5}$$

Since $x_2^{(5)} > x_2^{(4)} > x_1^{(3)}$ by (7.2), we have $P_3(x_2^{(5)}) > 0$ by Proposition 8.1, and consequently $P_7(x_2^{(5)}) < 0$ by Proposition 8.3. From (7.2) we have $x_1^{(7)} < x_2^{(5)} < x_3^{(7)}$. Proposition 8.1 implies that $P_7(x) < 0$ for $x \in (x_2^{(7)}, x_3^{(7)})$ and $P_7(x) > 0$ for $x \in (x_1^{(7)}, x_2^{(7)})$. Hence $x_2^{(5)} \in (x_2^{(7)}, x_3^{(7)})$. Thus

$$x_2^{(7)} < x_2^{(5)}. \tag{8.6}$$

Since $x_0^{(3)} < x_1^{(5)} < x_1^{(3)}$ (cf. (8.1)) we have $P_3(x_1^{(5)}) < 0$ by Proposition 8.1, hence $P_7(x_1^{(5)}) > 0$ by Proposition 8.3. As above we have $P_7(x) < 0$ for $x \in (x_2^{(7)}, x_3^{(7)})$ and for $x \in (x_0^{(7)}, x_1^{(7)})$. Hence $x_1^{(5)} \in (x_1^{(7)}, x_2^{(7)})$. Thus

$$x_1^{(7)} < x_1^{(5)}. \tag{8.7}$$

Arguing in the same way, repeatedly using the results of Propositions 8.1 and 8.3 and the already obtained results, we find by induction that the first line of (8.4) is satisfied for all m and $p = 1, 2, \ldots, m$.

Similarly we prove that the second line in (8.4) is satisfied for all m and $p = -1, -2, -3, \ldots, -m$.

■

9. Support points and convergence of zero sequences

In this section we assume that all the conditions and hence all the results of Section 6, 7, and 8 are satisfied.

Theorem 9.1. *Consider an indeterminate rational Stieltjes moment problem where the recurrence coefficients satisfy (7.3) and (7.4). Then for each $k = \pm 1, \pm 2, \ldots$, the zero sequence $\{x_k^{(2m)}\}_m$ converges to the zero $x_k^{(0)}$ of Φ_0 and the zero sequence $\{x_k^{(2m+1)}\}_m$ converges to the zero $x_k^{(\infty)}$ of Φ_∞.*

PROOF. We first consider $k > 0$. The sequence $\{x_k^{(2m+1)}\}_{m=k}$ is decreasing by Theorem 8.4 and is bounded below by x_0, hence converges to a limit y_k. It follows from Proposition 7.1 and Proposition 7.3 that every $x_p^{(\infty)}$ must coincide with at least one of the values y_k and every y_k must coincide with a value $x_p^{(\infty)}$.

The zeros $x_k^{(2m+1)}$ are simple zeros of Φ_{2m+1} and the zeros $x_p^{(\infty)}$ are simple zeros of Φ_∞. Consequently each $x_p^{(\infty)}$ is the limit of exactly one sequence $\{x_k^{(2m+1)}\}_m$, i.e., every $x_p^{(\infty)}$ coincides with exactly one y_k, and *vice versa*. By the ordering of the zeros $x_k^{(2m+1)}$ and the ordering of the zeros $x_p^{(\infty)}$ (cf. (7.6)) it follows that $y_k = x_k^{(\infty)}$ for $k = 1,2,\ldots$.

In the same way we find that $\{x_k^{(2m+1)}\}_{m\geq -k}$ converges to $x_k^{(\infty)}$ for $k = -1,-2,\ldots$.

Finally, Proposition 7.3 implies that the accumulation points of the set $\cup_{m=1}^{\infty} Z_{2m}$ in $(0,\infty)$ are exactly the zeros $x_k^{(0)}$ of Φ_0. Since $x_{k-1}^{(2m+1)} < x_k^{(2m)} < x_k^{(2m+1)}$ for $k = 1,2,\ldots$, all the accumulation points of $\{x_k^{(2m)}\}_m$ lie in $[x_{k-1}^{(\infty)}, x_k^{(\infty)}]$. The only zero of Φ_0 in $[x_{k-1}^{(\infty)}, x_k^{(\infty)}]$ is $x_k^{(0)}$, hence $\{x_k^{(2m)}\}_m$ converges to $x_k^{(0)}$. In the same way we establish that $\{x_k^{(2m)}\}_m$ converges to $x_k^{(0)}$ for $k = -1,-2,\ldots$.

■

Remark 9.2. It follows from Theorem 9.1 that supp μ_0 and supp μ_∞ consist of infinitely many points to the left of x_0 and to the right of x_0. Since Φ_0 and Φ_∞ are holomorphic in an open set containing $(0,\infty)$, we conclude that $x_k^{(0)}$ and $x_k^{(\infty)}$ tend to 0 as k tends to $-\infty$ and $x_k^{(0)}$ and $x_k^{(\infty)}$ tend to ∞ as k tends to ∞. In particular, the origin belongs to the support of both μ_0 and μ_∞.

When $\{\alpha_n\}$ tends monotonically to 0 and $\{\beta_n\}$ tends monotonically to $-\infty$, then the fact that the support accumulates at the origin and at infinity follows without the special assumptions in Theorem 6.3 from a Carleman-type criterion giving sufficient condition for determinacy. See [6, Theorem 6.2], [8, Theorem 16.2].

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